FINAL EXAM (BERGMAN) - ANSWER KEY

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(1) (a) Characteristic polynomial: $\lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\lambda + 2)^2$

Eigenvalues: $\lambda = 1, -2$

Eigenvectors:

$$\underline{\lambda = 1} : Span\left\{\begin{bmatrix}1\\1\\3\end{bmatrix}\right\}$$

$$\underline{\lambda = -2} : Span\left\{\begin{bmatrix}1\\-2\\3\end{bmatrix}\right\}$$

(b)

$$\mathbf{x}(t) = Ae^t\begin{bmatrix}1\\1\\3\end{bmatrix} + Be^{-2t}\begin{bmatrix}1\\-2\\3\end{bmatrix} + C\left(te^{-2t}\begin{bmatrix}1\\-2\\3\end{bmatrix} + e^{-2t}\begin{bmatrix}-\frac{1}{3}\\\frac{5}{3}\\1\end{bmatrix}\right)$$

Note: For the last vector, use generalized eigenvectors!

(c) Define $\boxed{w = u'}$, then we get:

$$\begin{cases} u' = & w \\ w' = & u'' = -u' - u + v = -u + w + v \\ v' = & 3u' + 6u - 2v = 6u + 3w - 2v \end{cases} \Rightarrow \begin{cases} u' = & w \\ w' = & -u + w + v \\ v' = & 6u + 3w - 2v \end{cases}$$

which gives us the system in (a).

(d)

$$\mathbf{x}(t) = \frac{1}{4}e^t\begin{bmatrix}1\\1\\3\end{bmatrix} - \frac{1}{6}e^{-2t}\begin{bmatrix}1\\-2\\3\end{bmatrix} + \frac{1}{4}\left(te^{-2t}\begin{bmatrix}1\\-2\\3\end{bmatrix} + e^{-2t}\begin{bmatrix}-\frac{1}{3}\\\frac{5}{3}\\1\end{bmatrix}\right)$$

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(2) (a)

$$W(f,g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t)$$

(b) Suppose f and g are linearly dependent.

Then there are constants a and b such that $af(t) + bg(t) = 0$ for all t

Now differentiating this, we get $af'(t) + bg'(t) = 0$ for all t

In other words, the following equation has a solution for every t:

$$\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But this means that (for every t), the above matrix is not invertible, hence its determinant is $= 0$ (for every t). That is, for every t:

$$f(t)g'(t) - f'(t)g(t) = 0$$

Thus, for every t:

$$W(f,g) = 0$$

(3) (a)

$$\begin{cases} b''(y) = & \lambda b(y) \\ a'(x) = & \lambda p(x)a(x) \end{cases}$$

(b)

$$u(x,y) = e^{\frac{-x^2}{2}} \cos(y)$$

(just choose $\lambda = -1$ and solve the corresponding equation! Also, you're allowed to set your constants to whatever you want them to be!)

(4)

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}$$

(5) Ignore this question, it is outside of the scope of this course!

(6) (a)

$$b_n = \frac{4}{\pi m}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right)$$

(b)

$$u(x,t) = \sum_{m=1}^{\infty} b_m e^{-\alpha\pi m t}\sin(\pi m x) = \sum_{m=1}^{\infty}\left(\frac{4}{\pi m}\left(1 - \cos\left(\frac{\pi n}{2}\right)\right)\right)e^{-\alpha\pi m t}\sin(\pi m x)$$

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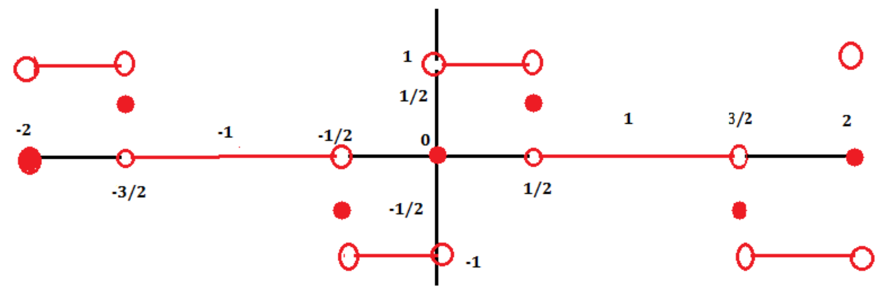

(c)